### 2.4.3 Definition (Boolean function)

Let (B, $\vee$, $\wedge$, $'$, 0, 1) be a boolean algebra. A function from $B^n$ to B called **boolean function** it can be specified by a boolean expression of $n$ variables.

### 2.4.4 Definition (Minterm or complete product or a Fundamental product)

A Boolean expression of $n$ variables $x_1, x_2, \ldots\ldots, x_n$ is said to be **minterm** or **complete product** or a **fundamental product** of $n$ variables if it is of the forms $\tilde{x}_1 \wedge \tilde{x}_2 \wedge \ldots\ldots \wedge \tilde{x}_n$, where $\tilde{x}_i$ denotes either $x_i$ or $x_i'$.

Observe that each minterm is completely determined by a sequence of 0's and 1's of length $n$, and any such sequence determines a number between 0 and $2^n - 1$ in binary representation.

A particular minterm will be denoted by $\min_j$ or $m_j$ if the associated sequence of its exponent gives the number $j$ in binary representation (Here $0 \le j \le 2^n - 1$). Thus, we have $2^n$ minterms in $n$ variables denoted by $m_0, m_1, \ldots\ldots, m_{2^n - 1}$. For example, in three variables $m_5 = x_1 \wedge x_2' \wedge x_3$, becomes in the binary representation 1 0 1.

Also, these minterms satisfy the following fundamental properties

(i) $m_i \wedge m_j = 0$ if $i \neq j$

(ii) $\bigvee_{i=0}^{2^n - 1} = m_0 \vee m_1 \vee \ldots\ldots \vee m_{2^n - 1} = 1$.

**2.4.5 Definition (Maxterm)**

A Boolean expression of $n$ variables $x_1, x_2, \ldots\ldots, x_n$ is said to be **maxterm** if it is of the form $\tilde{x}_1 \vee \tilde{x}_2 \vee \ldots\ldots \vee \tilde{x}_n$, when $\tilde{x}_i$ denotes either $\tilde{x}_i$ or $x_i'$.

Similarly the maxterm satisfy the following fundamentals properties :

(*i*) $M_i \vee M_j = 1$ if $i \neq j$

(*ii*) $\bigwedge_{i=1}^{2^n - 1} M_i = M_1 \wedge M_2 \wedge \ldots\ldots \wedge M_{2^n - 1} = 0$.

There are $2^n$ maxterm in $n$ variables denoted by $M_0, M_1, \ldots\ldots, M_{2^n - 1}$.

**2.4.6 Definition. (Disjunctive Normal form or Sum of Product or SOP) or DNP**

A boolean expression over two-valued Boolean algebra $(\{0, 1\}, \vee, \wedge, ', 0, 1)$ is said to be in disjunctive normal form (or sum of Product) if it is join of minterms.

**For example :** $(x_1' \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3)$ or

$x_1' x_2' x_3 + x_1' x_2' x_3' + x_1' x_2 x_3 = \Sigma\, m\,(1, 0, 3)$

is a boolean expression in disjunctive normal form of three minterms.

**2.4.7 Definition (Conjunctive normal form or Product of Sum or POS) CNP**

A boolean expression over two-value Boolean algebra $(\{0, 1\}, \vee, \wedge, ', 0, 1)$ is said to be in conjunctive normal form (or Product of sum) if it is meet of maxterms.

**For example.** $(x_1 \vee x_2' \vee x_3') \wedge (x_1' \vee x_2' \vee x_3') \wedge (x_1' \vee x_2' \vee x_3')$ Or

$(x_1 + x_2' + x_3')\,(x_1 + x_2 + x_3')\,(x_1' + x_2' + x_3') = \Pi\, M\,(3, 1, 7)$

is a boolean expression in conjunctive normal form of three variables.

**2.4.8 Obtaining Boolean expression in Disjunctive Normal form and conjunctive normal form**

(1) A Boolean expression can be obtained in Min term normal form **disjunctive normal form** corresponding to this function by having a **minterm** corresponding to each ordered $n$-table of 0 and 1 for which the value of the function is 1.

(2) A Boolean expression can be obtained in **conjunctive normal form** corresponding to this function by having a **maxterm** corresponding to 0 and 1 at which the value of function is **0**.

**Remark :** Boolean function represented as a sum of minterms or product of maxterms are said to be in canonical form.

**Example 1.** Simplify the Boolean expression

$$f(x, y, z) = (x' \wedge z) \vee (y \wedge z) \vee (y \wedge z')$$

and write in minterm normal form.

**Sol.**

$$f(x, y, z) = (x' \wedge z) \vee (y \wedge z) \vee (y \wedge z')$$

$$= (x' \wedge z) \vee [y \wedge (z \vee z')] \qquad \text{(Using distributive law)}$$

$$= (x' \wedge z) \vee (y \wedge 1) \qquad [\because z \vee z' = 1]$$

$$= (x' \wedge z) \vee y$$

| $x$ | $y$ | $z$ | $x'$ | $x' \wedge z$ | $f = (x' \wedge z) \vee y$ | min |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | $m_0$ |
| 0 | 0 | 1 | 1 | 1 | 1 | $m_1$ |
| 0 | 1 | 0 | 1 | 0 | 1 | $m_2$ |
| 0 | 1 | 1 | 1 | 1 | 1 | $m_3$ |
| 1 | 0 | 0 | 0 | 0 | 0 | $m_4$ |
| 1 | 0 | 1 | 0 | 0 | 0 | $m_5$ |
| 1 | 1 | 0 | 0 | 0 | 1 | $m_6$ |
| 1 | 1 | 1 | 0 | 0 | 1 | $m_7$ |

Since minterms corresponds to each ordered triple of 0 and 1 for which the value of the function is 1.

$\therefore$ minterm are $\quad m_1 = x' \wedge y' \wedge z,\ m_2 = x' \wedge y \wedge z',\ m_3 = x' \wedge y \wedge z.$

$m_6 = x \wedge y \wedge z',\ m_7 = x \wedge y \wedge z$

Hence Minterm Normal form

$$= m_1 \vee m_2 \vee m_3 \vee m_6 \vee m_7$$

$$= (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$$

**Example 2.** Simplify the Boolean expression

$$f(x, y, z) = (x \wedge y' \wedge z) \vee (x \wedge y \wedge z)$$

and find its conjective normal forms.

**Sol.**

$$f(x, y, z) = (x \wedge y' \wedge z) \vee (x \wedge y \wedge z)$$

$$= (x \wedge z \wedge y') \vee (x \wedge z \wedge y)$$

$$= [(x \wedge z) \wedge (y' \vee y)]$$

$= [(x \wedge z) \wedge 1]$

$= x \wedge z$

| $x$ | $y$ | $z$ | $f = x \wedge z$ | Max |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $M_0$ |
| 0 | 0 | 1 | 0 | $M_1$ |
| 0 | 1 | 0 | 0 | $M_2$ |
| 0 | 1 | 1 | 0 | $M_3$ |
| 1 | 0 | 0 | 0 | $M_4$ |
| 1 | 0 | 1 | 1 | $M_5$ |
| 1 | 1 | 0 | 0 | $M_6$ |
| 1 | 1 | 1 | 1 | $M_7$ |

Since Maxterm corresponds to each ordered triple of 0 and 1 for which the value of the function is 0.

$x \vee y \vee z$ $\quad x \vee y \vee z'$ $\quad x \vee y' \vee z$

$\therefore$ Maxterm are $M_0 = x' \vee y' \vee z'$, $M_1 = x' \vee y' \vee z$, $M_2 = x' \vee y \vee z'$, $M_3 =$

$x \vee y' \vee z'$ $\quad x' \vee y \vee z$

$x' \vee y \vee z$, $M_4 = x \vee y' \vee z'$, $M_6 = x \vee y \vee z'$. $M_6 = x' \vee y' \vee z$

Hence ~~disjunctive~~ conjuctive normal form

$= M_0 \wedge M_1 \wedge M_2 \wedge M_3 \wedge M_4 \wedge M_6 = (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \vee (x \vee y' \vee z') \wedge (x' \vee y \vee z) \wedge (x' \vee y' \vee z)$

$= \cancel{(x' \vee y' \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y \vee z') \wedge (x' \vee y \vee z) \wedge (x \vee y' \vee z') \wedge}$

$\cancel{(x \vee y \vee z')}$

**2.4.9 Algorithm for obtaining complete Sum-of-Product Expression**

Let the given boolean expression be $f(x_1, x_2, \ldots\ldots, x_n)$

**Step 1.** Find a product P in $f(x_1, x_2, \ldots\ldots, x_n)$ which does not contain the variable $x_i$ and then multiply P by $(x_i + x_i')$, deleting any repeated products (as $x + x' = 1$ and $P + P = P$)

**Step 2.** Repeat Step 1 until every product in $f(x_1, x_2, \ldots\ldots, x_n)$ is a minterm i.e. every product P contains all the $n$-variables.

**2.4.10 Algorithm for obtaining Product of sum canonical form**

Let the given boolean expression be $f(x_1, x_2, \ldots\ldots, x_n)$

**Step 1.** Find a sum S in $f(x_1, x_2, \ldots\ldots, x_n)$ which does not contains the variable $x_i$ and then add S by $(x_i\ x_i')$, deleting any repeated sum (as $xx' = 0$ and $SS = S$)

**Step 2.** Repeat Step 1 till every sum in $f(x_1, x_2, \ldots\ldots, x_n)$ is a maxterm i.e. every sum S contains all the $n$-variables.

**Example 3.** Using Boolean algebra, construct the DNF of the boolean function

$$f(x, y, z) = x(y + z)$$

**Sol.** Here

$$\begin{aligned} f(x, y, z) &= x(y + z) = xy + xz \\ &= xy \cdot 1 + xz \cdot 1 \\ &= xy(z + z') + xz(y + y') \\ &= xyz + xyz' + xyz + xy'z \\ &= (xyz + xyz) + xy'z + xyz' \\ &= xyz + xy'z + xyz' \end{aligned}$$

which is in the DNF of the boolean function $f(x, y, z)$.

**Example 4.** Express $x_1 + x_2$ and $x_1x_2$ in its complete Sum-of-Product term in three variables $x_1, x_2, x_3$.

**Sol.** (*i*) Now

$$\begin{aligned} x_1 + x_2 &= x_1 \cdot 1 + x_2 \cdot 1 = x_1(x_2 + x_2') + x_2(x_1 + x_1') \\ &= x_1x_2 + x_1x_2' + x_1x_2 + x_1'x_2 \\ &= x_1x_2 \cdot 1 + x_1x_2' \cdot 1 + x_1'x_2 \cdot 1 \\ &= x_1x_2(x_3 + x_3') + x_1x_2'(x_3 + x_3') + x_1'x_2(x_3 + x_3') \\ &= x_1x_2x_3 + x_1x_2x_3' + x_1x_2'x_3 + x_1x_2'x_3' + x_1'x_2x_3 + x_1'x_2x_3' \\ &= x_1x_2x_3 + x_1x_2x_3' + x_1x_2'x_3 + x_1'x_2x_3 + x_1'x_2x_3' + x_1x_2'x_3' \end{aligned}$$

Which is the complete Sum-of-Product form.

(*ii*) Also,

$$\begin{aligned} x_1x_2 &= x_1x_2 \cdot 1 = x_1x_2(x_3 + x_3') \\ &= x_1x_2x_3 + x_1x_2x_3' \end{aligned}$$

which is the complete Sum-of-Product form.

**Example 5.** Obtain Product of Sum Canonical form of boolean expression $x_1x_2$ in three variables $x_1, x_2, x_3$.

**Sol.** Here

$$\begin{aligned} x_1x_2 &= (x_1 + 0)(x_2 + 0) \\ &= [x_1 + (x_2 x_2')][x_2 + (x_1 x_1')] && xx' = 0 \\ &= (x_1 + x_2)(x_1 + x_2')(x_2 + x_1)(x_2 + x_1') && [\because xx = x] \\ &= (x_1 + x_2)(x_1 + x_2')(x_2 + x_1') \end{aligned}$$

$$= [(x_1 + x_2) + (x_3 x_3')] [(x_1 + x_2') + (x_3 x_3')] [(x_2 + x_1') + (x_3 x_3')]$$

$$= (x_1 + x_2 + x_3)(x_1 + x_2 + x_3')(x_1 + x_2' + x_3)(x_1 + x_2' + x_3')(x_2 + x_1' + x_3)(x_2 + x_1' + x_3')$$

which is the required product of Sum Canonical form.

**Example 6.** Show that $(x_1' x_2' x_3' x_4') + (x_1' x_2' x_3' x_4) + (x_1' x_2' x_3 x_4) + (x_1' x_2 x_3 x_4) = x_1' x_2'$.

**Sol.**

$$\begin{aligned}
\text{L.H.S.} &= (x_1' x_2' x_3' x_4') + (x_1' x_2' x_3' x_4) + (x_1' x_2' x_3 x_4) + (x_1' x_2 x_3 x_4') \\
&= [(x_1' x_2' x_3' x_4') + (x_1' x_2' x_3' x_4)] + [(x_1' x_2' x_3 x_4) + (x_1' x_2' x_3 x_4')] \\
&= x_1' x_2' x_3' (x_4' + x_4) + x' x_2' x_3 (x_4 + x_4') \\
&= x_1' x_2' x_3' \cdot 1 + x_1' x_2' x_3 \cdot 1 \\
&= x_1' x_2' (x_3' + x_3) \\
&= x_1' x_2' \cdot 1 \\
&= x_1' x_2' \\
&= \text{R.H.S.}
\end{aligned}$$

**Example 7.** Show that the following Boolean expressions are equivalent to one another. Obtain their Sum-of-Product Canonical form

(*i*) $f_1(x, y, z) = (x + y)(x' + z)(y + z)$ (*ii*) $f_2(x, y, z) = (x z) + (x' y) + (y z)$

(*iii*) $f_3(x, y, z) = (x + y)(x' + z)$ (*iv*) $f_4(x, y, z) = xz + x'y$

**Sol.** The binary valuation of the given boolean expression are

| $x$ | $y$ | $z$ | $x+y$ | $x'+z$ | $y+z$ | $f_1$ | $f_3$ | $xz$ | $x'y$ | $yz$ | $f_2$ | $f_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |

Since the values of the boolean expressions for $f_1, f_2, f_3$ and $f_4$ are equal over every triple of the two-value element Boolean algebra. So these are equivalent.

To write them in Sum-of-Product cannonical form.

$\therefore$ We have $\quad f_4(x, y, z) = xz + x'y = xz \cdot 1 + x'y \cdot 1$

$$= xz\,(y + y') + x'y\,(z + z')$$

$$= xyz + xy'z + x'yz + x'yz'$$

which is in the Sum-of-Product canonical form.

**Example 8.** Obtain the Sum-of-Products Canonical forms of the following Boolean expression

(*a*) $x_1 + x_2$ (*b*) $x_1 + (x_2\, x_3')$ (*c*) $(x_1\, x_2') + x_4$

assuming that this is an expression in four variables $x_1, x_2, x_3$ and $x_4$.

**Sol.** The Sum-of-Product canonical form of the given expression can be obtained from the following table.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_1 + x_2$ | $\text{Min}_j$ | $x_3'$ | $x_2 x_3'$ | $x_1 + (x_2\, x_3')$ | $\text{Min}'_j$ | $x_2'$ | $x_1 x_2'$ | $x_1 x_2' + x_4$ | $\text{Min}''_j$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | $m_0$ | 1 | 0 | 0 | $m'_0$ | 1 | 0 | 0 | $m''_0$ |
| 0 | 0 | 0 | 1 | 0 | $m_1$ | 1 | 0 | 0 | $m'_1$ | 1 | 0 | 1 | $m''_1$ |
| 0 | 0 | 1 | 0 | 0 | $m_2$ | 0 | 0 | 0 | $m'_2$ | 1 | 0 | 0 | $m''_2$ |
| 0 | 0 | 1 | 1 | 0 | $m_3$ | 0 | 0 | 0 | $m'_3$ | 1 | 0 | 1 | $m''_3$ |
| 0 | 1 | 0 | 0 | 1 | $m_4$ | 1 | 1 | 1 | $m'_4$ | 0 | 0 | 0 | $m''_4$ |
| 0 | 1 | 0 | 1 | 1 | $m_5$ | 1 | 1 | 1 | $m'_5$ | 0 | 0 | 1 | $m''_5$ |
| 0 | 1 | 1 | 0 | 1 | $m_6$ | 0 | 0 | 0 | $m'_6$ | 0 | 0 | 0 | $m''_6$ |
| 0 | 1 | 1 | 1 | 1 | $m_7$ | 0 | 0 | 0 | $m'_7$ | 0 | 0 | 1 | $m''_7$ |
| 1 | 0 | 0 | 0 | 1 | $m_8$ | 1 | 0 | 1 | $m'_8$ | 1 | 1 | 1 | $m''_8$ |
| 1 | 0 | 0 | 1 | 1 | $m_9$ | 1 | 0 | 1 | $m'_9$ | 1 | 1 | 1 | $m''_9$ |
| 1 | 0 | 1 | 0 | 1 | $m_{10}$ | 0 | 0 | 1 | $m'_{10}$ | 1 | 1 | 1 | $m''_{10}$ |
| 1 | 0 | 1 | 1 | 1 | $m_{11}$ | 0 | 0 | 1 | $m'_{11}$ | 1 | 1 | 1 | $m''_{11}$ |
| 1 | 1 | 0 | 0 | 1 | $m_{12}$ | 1 | 1 | 1 | $m'_{12}$ | 0 | 0 | 0 | $m''_{12}$ |
| 1 | 1 | 0 | 1 | 1 | $m_{13}$ | 1 | 1 | 1 | $m'_{13}$ | 0 | 0 | 1 | $m''_{13}$ |
| 1 | 1 | 1 | 0 | 1 | $m_{14}$ | 0 | 0 | 1 | $m'_{14}$ | 0 | 0 | 0 | $m''_{14}$ |
| 1 | 1 | 1 | 1 | 1 | $m_{15}$ | 0 | 0 | 1 | $m'_{15}$ | 0 | 0 | 1 | $m''_{15}$ |

(*a*) $x_1 + x_2 = m_4 + m_5 + m_6 + m_7 + m_8 + m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15}$

$\cancel{= (x_1' x_2 x_3' x_4') + (x_1' x_2 x_3' x_4) + (x_1' x_2 x_3 x_4') + (x_1' x_2 x_3 x_4) +}$

$= x_1'x_2x_3'x_4' + x_1'x_2x_3'x_4 + x_1'x_2x_3x_4' + x_1'x_2x_3x_4 + x_1x_2'x_3'x_4' + x_1x_2'x_3'x_4$

$\cancel{(x_1 x_2' x_3' x_4') + (x_1 x_2' x_3' x_4) + (x_1 x_2 x_3' x_4') + (x_1 x_2 x_3' x_4)}$

$+ x_1x_2'x_3x_4' + x_1x_2'x_3x_4 + x_1x_2x_3'x_4' + x_1x_2x_3'x_4 + x_1x_2x_3x_4' + x_1x_2x_3x_4$

(*b*) $x_1 + (x_2 x_3') = m_4' + m_5' +$ [illegible] $+ m_8' + m_9' + m_{10}' + m_{11}' + m_{12}' + m_{13}' +$

$m_{14}' + m_{15}'$

$= (x_1' x_2 x_3' x_4') + (x_1' x_2 x_3' x_4) + (x_1' x_2' x_3' x_4') + (x_1 x_2' x_3' x_4') + (x_1 x_2' x_3' x_4') +$

$(x_1 x_2' x_3' x_4') + (x_1' x_2 x_3' x_4') + (x_1 x_2 x_3' x_4') + (x_1 x_2 x_3 x_4')$

$\rightarrow m_1'' + m_3'' + m_5'' + m_7'' + m_8'' + m_9'' + m_{10}'' + m_{11}'' + m_{13}'' + m_{15}''$

(*c*) $(x_1 x_2') + x_4 = x_1' x_2' x_3' x_4 + x_1' x_2' x_3 x_4 + x_1' x_2 x_3' x_4 + x_1' x_2 x_3 x_4 + x_1 x_2' x_3' x_4'$

$x_1 x_2' x_3' x_4 + x_1 x_2' x_3 x_4' + x_1 x_2' x_3 x_4 + x_1 x_2 x_3' x_4 + x_1 x_2 x_3 x_4$

**Example 9.** Find the Boolean Expression that defines the function $f$ by

| | |
|---|---|
| $f(0, 0, 0) = 0$ | $f(0, 0, 1) = 0$ |
| $f(1, 0, 0) = 1$ | $f(1, 1, 0) = 0$ |
| $f(0, 1, 0) = 1$ | $f(0, 1, 1) = 0$ |
| $f(1, 0, 1) = 1$ | $f(1, 1, 1) = 1$ |

**Sol.** The minterms are $f(0,1,0)$, $f(1,0,0)$, $f(1,0,1)$, $f(1,1,1)$

i.e. $(x' \wedge y \wedge z')$, $(x \wedge y' \wedge z')$, $(x \wedge y' \wedge z)$, $(x \wedge y \wedge z)$

D.N.F is

$$f(x, y, z) = (x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z)$$

can be simplified as

$= (x' \wedge y \wedge z') \vee x \wedge [(y' \wedge z') \vee (y' \wedge z) \vee (y \wedge z)]$ [Distributive Law]

$= (x' \wedge y \wedge z') \vee x \wedge [\{y' \wedge (z' \vee z)\} \vee (y \wedge z)]$ [Distributive Law]

$= (x' \wedge y \wedge z') \vee x \wedge (y' \vee (y \wedge z))$ $[\because z' \vee z = 1,\ y' \wedge 1 = y']$

$= (x' \wedge y \wedge z') \vee x \wedge (y' \vee y) \wedge (y' \vee z)$ [Distributive Law]

$= (x' \wedge y \wedge z') \vee x \wedge (y' \vee z)$ $[\because y' \vee y = 1]$

$= (x' \wedge y \wedge z') \vee [(x \wedge y') \vee (x \wedge z)]$ [Distributive Law]

**Example 10.** Find the Boolean Expression in CN-form.

| $x$ | $y$ | $z$ | $f$ | |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | $M_0$ $(x \vee y \vee z)$ |
| 0 | 0 | 1 | 0 | $M_1 = (x \vee y \vee z')$ |
| 0 | 1 | 0 | 1 | $M_2 = (x \vee y' \vee z)$ |
| 0 | 1 | 1 | 0 | $M_3 = (x \vee y' \vee z')$ |
| 1 | 0 | 0 | 0 | $M_4 = (x' \vee y \vee z)$ |
| 1 | 0 | 1 | 0 | $M_5 = (x' \vee y \vee z')$ |
| 1 | 1 | 0 | 0 | $M_6 = (x' \vee y' \vee z)$ |
| 1 | 1 | 1 | 1 | $M_7 = (x' \vee y' \vee z')$ |

CN form is $M_0 \wedge M_3 \wedge M_4 \wedge M_5 \wedge M_6$

$(x \vee y \vee z') \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z) \wedge \cancel{(x' \vee y \vee z')} \wedge (x' \vee y' \vee z)$.

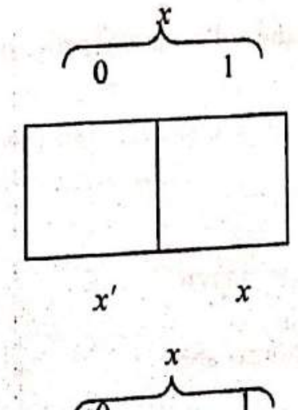
### 2.4.11 Karnaugh Maps (or K-maps)

A Karnaugh map structure is an area which is subdivided into $2^n$ cells, one for each possible output combination for a Boolean function of $n$ variables. Half the number of cells are associated with an input value of 1 for one of the variables and the other half the number of cells, with the input value 0 for the same variable More precisely, the K-map corresponds to Boolean expression in variable is an area which is subdivided into $2^n$ cells (sequences) each of which corresponds to one of the fundamental products or minterms in n-variables.

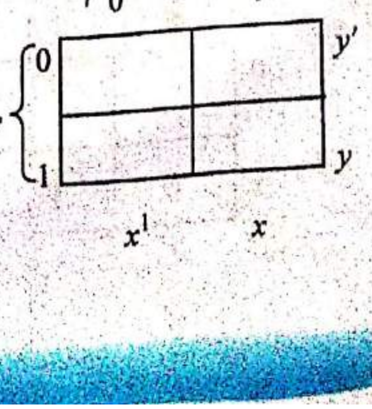
For example, K-map in three variables has $2^3$ (= 8) cells each of which correspond to.one of the following minterm $xyz$, $xyz'$, $xy'z$, $xy'z'$, $x'yz$, $x'yz'$, $x'y'z$. $x'y'z'$.

K-maps for 4 different variables is shown below ;

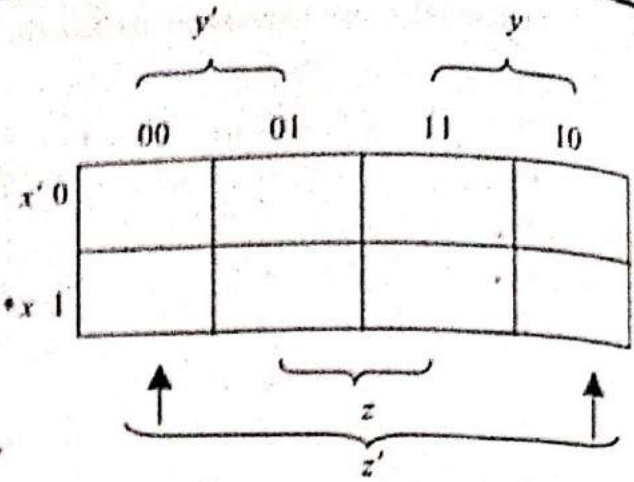
(*a*) K-map for 1 variable



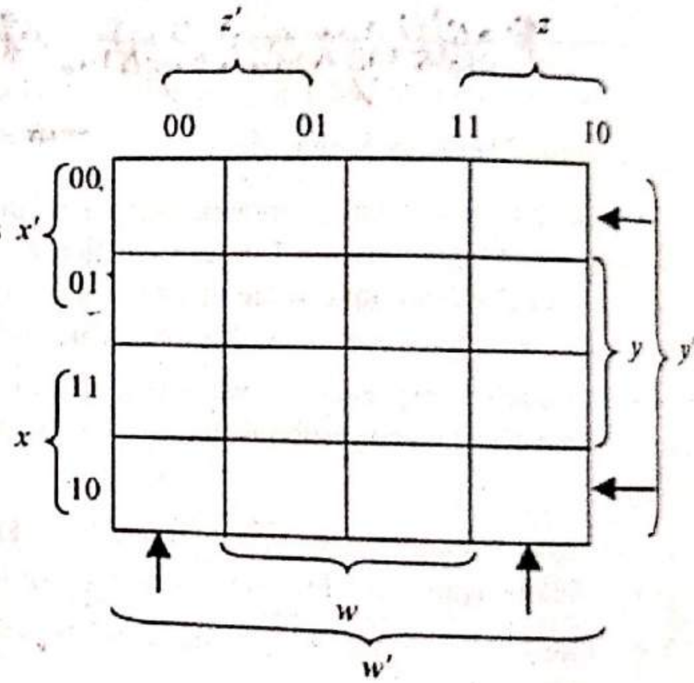

(*b*) K-map for 2 variable $y$



(c) K-map for 3 variables




(d) K-map for 4 variables




**Example 1.** Find the K-map for the following expression

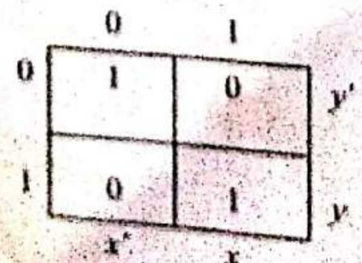
(a) $xy + x'y'$

(b) $x'y'z + x'yz' + xyz'$

(c) $x'y'zw + x'yzw' + xy'zw + xyzw'$

**Sol.** K-maps for the given expressions are

(a)



(b)

| | | $y'$ | | $y$ | |
|---|---|---|---|---|---|
| | | 00 | 01 | 11 | 10 |
| $x'$ | 0 | 0 | 1 | 0 | 1 |
| $x$ | 1 | 0 | 0 | 0 | 1 |

$z$ (under columns 01, 11); $z'$ (under columns 00, 10)

$(\text{cell})_{11} = 000$ no term in expression (b) as $x'y'z'$ & output is 0
$(\text{cell})_{12} = 001$, There is term $x'y'z$ in expression & value of output is 1
$(\text{cell})_{24} = 110$, term $xyz'$ exists in the expression & value is 1

(c)

| | | $z'$ | | $z$ | | |
|---|---|---|---|---|---|---|
| | | 00 | 01 | 11 | 10 | |
| $x'$ | 00 | 0 | 0 | 1 | 0 | $y'$ |
| | 01 | 0 | 0 | 0 | 1 | $y$ |
| $x$ | 11 | 0 | 0 | 0 | 1 | $y$ |
| | 10 | 0 | 0 | 1 | 0 | $y'$ |

$w$ (under columns 01, 11); $w'$ (under columns 00, 10)

1st cell = 0000 no term in expression having $x'y'z'w'$ & value is 0
$(\text{cell})_{12} = 0001 =$ no term in expression having $x'y'z'w$ & output 0
$(\text{cell})_{13} = 0011 =$ there is term $x'y'zw$ in expression & value is 1
$(\text{cell})_{24} = 0110$, There is term in the given expression as $x'yzw'$ & output value is 1 & so on
Find the other outputs

**Example 2.** Find the Boolean expression represented by the following truth table, given K-map representation. Also write the expression.

| $x$ | $y$ | $z$ | $f(x, y, z)$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

minterms are $f(000)$, $f(001)$, $f(1,0,0)$ & $f(110)$

D.N.F is

$f(x,y,z) = f(000) \vee f(001) \vee f(1,0,0) \vee f(110)$

$= (x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z')$

**Sol.** The following figure represents the K-map for the given Boolean expression.

| | $y'$ | | $y$ | |
|---|---|---|---|---|
| | 00 | 01 | 11 | 10 |
| $x'$ 0 | 1 | 1 | 0 | 0 |
| $x$ 1 | 1 | 0 | 0 | 1 |
| | $z'$ | $z$ | $z$ | $z'$ |

The expression $x'y'z' + x'y'z + xy'z' + xyz'$

# EXERCISE 2 (c)

1. Show that the Boolean functions $f_1(x, y, z) = (x_1 \vee x_2) \vee x_3$ and $f_2(x, y, z) = x_1 \vee (x_2 \vee x_3)$ are equivalent.

2. Construct the truth table for the following expressions

   (a) $f_1(x_1, x_2) = x_1 \vee x_2$ (b) $f_2(x_1, x_2) = x_1 \wedge x_2$ (c) $f_3(x_1) = x_1'$

3. Find the truth value of $f(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge (x_1' \vee x_2') \wedge (x_2 \vee x_3')'$

4. Write the following Boolean expressions in an equivalent sum of Product cannonical form in three variables $x_1, x_2\, x_3$

   (a) $x_1 \wedge x_2'$ (b) $x_2 \vee x_3'$ (c) $(x_1 \vee x_2)' \vee (x_1' \wedge x_3)$

5. Obtain the product of sums cannonical form in three variables $x_1, x_2, x_3$ for expressions.

   (a) $x_2 \vee x_3$ (b) $x_2 \wedge x_3$.

6. Find the value of the Boolean expression given below

   (a) $[x \wedge (y \vee (x \wedge \overline{y}))] \vee [(x \wedge \overline{y}) \vee \overline{(x \wedge \overline{z})}]$ for $x = 1, y = 1$ and $z = 0$

   (b) $x + \overline{yz}$ for $x = 0, y = 1, z = 1$.

7. Obtain the value of the Boolean forms

   (a) $x_1 \wedge (x_1' \vee x_2)$ (b) $x_1 \wedge x_2$ (c) $x_1 \vee (x_1 \wedge x_2)$

8. Obtain the sum of Products and Product-of-Sums cannonical forms of the following :

(a) $x_1 x_2' + x_3$ (b) $[(x_1 + x_2)(x_3 x_4)']'$

(c) $x_1' + [x_2' + x_1 + (x_2 x_3)'](x_2 + x_1' x_2)$

9. Find the Karnough map for each boolean expression.

(a) $xy' + x'y + x'y'$ (b) $xy' + x'y$

## ANSWERS

2.

| $x_1$ | $x_2$ | $f_1 = x_1 \vee x_2$ | $f_2 = x_1 \wedge x_2$ | $f_3 = x_1'$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 |

3.

| $x_1$ | $x_2$ | $x_3$ | $x_1 \vee x_2$ | $x_1' \vee x_2'$ | $(x_2 \vee x_3)'$ | $f(x_1, x_2, x_3)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 |

4. (a) $(x_1 \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_2')$

(b) $[(x_1 \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3)] \vee [(x_1 \wedge x_2 \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3')] \vee$
$[(x_1 \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2' \wedge x_3')]$

(c) $(x_1 \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3)$

5. (a) $(x_1 \vee x_2 \vee x_3) \wedge (x_1' \vee x_2 \vee x_3)$

(b) $(x_1 \vee x_2 \vee x_3) \wedge (x_1' \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3') \wedge (x_1' \vee x_2 \vee x_3') \wedge$
$(x_1 \vee x_2' \vee x_3) \wedge (x_1' \vee x_2' \vee x_3)$